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LETTER TO THE EDITOR

First integrals and Yoshida analysis of Nahm's equations

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Abstract. First integrals of Nahm's equations for monopole solutions in Yang-Mills theories are explicitly given for various special cases. Taking into account the scale invariance, all Kowalewski exponents are determined. Both the polynomial first integrals and the Kowalewski exponents are obtained with the help of a REDUCE program package. This leads to a completely consistent picture in the sense of the analysis recently proposed by Yoshida.

Nahm's equations arise in the construction of monopole solutions in Yang-Mills theories (Nahm 1982, Hitchin 1983, Donaldson 1984, Corrigan and Goddard 1984). Let T_i, $i=1,2,3$, be $n\times n$ matrices of a complex-valued function of the variable t. Then Nahm's equations are

$$\frac{\mathrm{d}T_i}{\mathrm{d}t}=\tfrac{1}{2}\varepsilon_{ijk}[T_j, T_k]. \tag{1}$$

They represent an autonomous system of $3n^2$ ordinary differential equations with quadratic non-linearity on the right-hand side. All indices range over 1, 2, 3, ε_{ijk} is the totally antisymmetric tensor with $\varepsilon_{123}=1$ and the Einstein summation convention is used throughout, except where stated otherwise. Equation (1) is scale invariant under the similarity transformations $\bar{t}=\alpha^{-1}t$, $\bar{T}_{j,kl}=\alpha T_{j,kl}$. Another remarkable property of equation (1) is the existence of a Lax representation (Rouhani 1984, Ward 1985).

For any system of autonomous equations it is important to know whether there exist first integrals and, if so, to determine them explicitly. There are two different methods of tackling this problem. On the one hand, an ansatz for first integrals within a well defined class of functions is made which contains a number of free parameters. If it is possible to adjust these parameters such that the total derivative with respect to time vanishes, a first integral has been found. In general, this method suffers from the fact that the necessary calculations increase tremendously with the size of the problem. For polynomial systems of equations and polynomial first integrals, however, there exists a REDUCE program package which does these calculations almost completely automatically (Schwarz 1986a). The second approach uses the well known fact that there is a close connection between the existence of algebraic first integrals and the analytic properties of the solution set.

For scale invariant systems this relation is much tighter than in general as shown recently by Yoshida (1983a, b). His main results may be stated as follows. Let

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$x = (x_1, \ldots, x_n)$ be n functions† depending on the variable t and let

$$\dot{x}_i = \omega_i(x_1, \ldots, x_n) \qquad (i = 1, \ldots, n) \tag{2}$$

be a system of n autonomous differential equations with rational functions ω_i. Assume that this system is scale invariant under the transformations $\bar{t} = \alpha^{-1}t$, $\bar{x}_i = \alpha^{d_i}x_i$, $i = 1, \ldots, n$. If the constants $c = (c_1, \ldots, c_n)$ are determined from the system of algebraic equations

$$-d_i c_i = \omega_i(c_1, \ldots, c_n) \qquad (i = 1, \ldots, n) \tag{3}$$

then the eigenvalues of the matrix

$$K_{ij} = \left.\frac{\partial \omega_i(x)}{\partial x_j}\right|_{x=c} + d_i \delta_{ij} \tag{4}$$

are called Kowalewski exponents. In the literature (compare Steeb *et al* (1985) and references therein) they are also called resonances. Yoshida (1983a, b) obtained the following relations between the Kowalewski exponents and the possible existence of first integrals of a scale invariant system (2).

(i) If $\Phi(x)$ is a homogeneous first integral of degree M and grad$\Phi(x)$ is not identically zero on the solution set of (3), then $\rho = M$ occurs as a Kowalewski exponent. Beyond that the scale parameters d_i may occur as eigenvalues of the matrix (4).

(ii) Let $\Phi(x)$ and $\Phi'(x)$ be two independent first integrals of (2) with the same degree M. Suppose the two vectors grad$\Phi(x)$ and grad$\Phi'(x)$ are both finite, not identically zero and linear independent for $x = c$, c a solution of equation (3), then $\rho = M$ occurs as a Kowalewski exponent with multiplicity not lower than 2.

(iii) In order that a given similarity invariant system (2) is algebraically integrable, it is necessary that every possible Kowalewski exponent be a rational number. In other words, if there exists at least one irrational or imaginary Kowalewski exponent, the system (2) is not algebraically integrable.

These results will be applied now to various special cases of Nahm's equations. Let us consider first the case $n = 2$ of equations (1) so that there are twelve equations and unknown functions. Applying the REDUCE package of Schwarz (1986a) the following first integrals are identified:

$$F_i = T_{i,11} + T_{i,22} \qquad i = 1, 2, 3 \tag{5}$$

$$G_1 = \sum_{j=1}^{3} T_{j,12}^2 \qquad G_2 = \sum_{j=1}^{3} T_{j,21}^2 \tag{6}$$

$$H_i = T_{i,12}T_{i,21} + \sum_{j=1}^{3} T_{j,11}T_{j,22} - T_{i,11}T_{i,22} \qquad i = 1, 2, 3. \tag{7}$$

The system of algebraic equations following from (3) has two independent non-trivial solutions. In both cases the Kowalewski determinant factorises completely over the integers into

$$\det(K_{ij} - \rho\delta_{ij}) = (\rho + 1)\rho^3(\rho - 1)^3(\rho - 2)^5. \tag{8}$$

This form of the Kowalewski determinant has been obtained by applying the REDUCE package KOWAL (Schwarz 1986b). The threefold zero $\rho = 1$ corresponds to the first-

† The constant n used in this paragraph has no relation whatsoever to the variable n used in connection with Nahm's equation above and subsequently.

order integrals (5) whereas the fivefold zero $\rho = 2$ corresponds to the second-order integrals (6) and (7). Thus we have a completely consistent picture obtained by two methods. Yoshida's theorem assures us that there are no polynomial integrals of higher order with non-vanishing gradient on the solutions (9) and (10).

The case $n = 2$ for the special choice $T_j(t) = if_j(t)\sigma_j$, $j = 1, 2, 3$, where σ_j are the Pauli matrices, has been studied before (Steeb *et al* 1983). It is obtained from the results described above if $T_{1,12} = T_{1,21} = f_1$, $T_{2,21} = -T_{2,12} = f_2$ and $T_{3,11} = -T_{3,22} = f_3$ are taken with the remaining entries identically zero. There is no first-order integral in this case whereas two independent second-order integrals are obtained, e.g., as $G_1 = f_1^2 - f_2^2$ and $H_1 = f_1^2 - f_3^2$. The Kowalewski exponents are $\rho = -1$ and $\rho = 2$ (twofold). The two first integrals correspond to this latter exponent.

Next we consider the case where the matrices T_i are linear combinations of the generators of $\mathrm{sl}(n)$. Let H_α, $\alpha = 1, \ldots, n-1$ be the generators of the Cartan subalgebra of $\mathrm{sl}(n)$ and let $\{E_\alpha, E_{-\alpha}\}$ be step operators satisfying the relations (Humphreys 1980)

$$[H_\alpha, E_{\pm\beta}] = \pm \boldsymbol{K}_{\alpha\beta} E_{\pm\beta} \qquad [E_\alpha, E_{-\beta}] = \delta_{\alpha\beta} H_\beta \qquad \text{(no summation)} \tag{9}$$

where $\{\boldsymbol{K}_{\alpha\beta}\}$ is the $(n-1)\times(n-1)$ Cartan matrix of $\mathrm{sl}(n)$. Assume that

$$T_1(t) = \tfrac{1}{2}\mathrm{i} \sum_{\alpha=1}^{n-1} q_\alpha(t)(E_\alpha + E_{-\alpha}) \tag{10}$$

$$T_2(t) = -\tfrac{1}{2} \sum_{\alpha=1}^{n-1} q_\alpha(t)(E_\alpha - E_{-\alpha}) \tag{11}$$

$$T_3(t) = \tfrac{1}{2}\mathrm{i} \sum_{\alpha=1}^{n-1} p_\alpha(t) H_\alpha . \tag{12}$$

Then Nahm's equations take the form

$$\dot{p}_\alpha = q_\alpha^2 \qquad \dot{q}_\alpha = \tfrac{1}{2} \sum_{\beta=1}^{n-1} p_\beta \boldsymbol{K}_{\alpha\beta} q_\alpha . \tag{13}$$

At first let $n = 2$. The Cartan matrix is equal to 2 and the system (13) reduces to the pair of equations $\dot{p} = q^2$, $\dot{q} = pq$. There is a single first integral of second order $h(p, q) \equiv p^2 - q^2$ with the Kowalewski exponent $\rho = 2$. For $n = 3$ the Cartan matrix is given by

$$\{\boldsymbol{K}_{\alpha\beta}\} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \tag{14}$$

with the corresponding equations of motion

$$\begin{aligned} &\dot{p}_1 = q_1^2 \qquad \dot{p}_2 = q_2^2 \\ &\dot{q}_1 = q_1(p_1 - \tfrac{1}{2}p_2) \qquad \dot{q}_2 = q_2(p_2 - \tfrac{1}{2}p_1). \end{aligned} \tag{15}$$

There are two independent first integrals of homogeneity degree two and three respectively:

$$\begin{aligned} h_1(\boldsymbol{p}, \boldsymbol{q}) &= p_1^2 + p_2^2 - p_1 p_2 - q_1^2 - q_2^2 \\ h_2(\boldsymbol{p}, \boldsymbol{q}) &= p_1^2 p_2 - p_1 p_2^2 + p_1 q_2^2 - p_2 q_1^2 . \end{aligned} \tag{16}$$

The algebraic system for the c_i ($i = 1, \ldots, 4$) which follows from equations (15) has five solutions. Only for one of them are all c_i different from zero, i.e. $c_1 = c_2 = -2$,

$c_3^2 = c_4^2 = 2$. Let us call this alternative the main branch. With these values for the c_i the following expression for the Kowalewski determinant is obtained:

$$\det(K_{ij} - \rho\delta_{ij}) = (\rho+2)(\rho+1)(\rho-2)(\rho-3). \tag{17}$$

The two first integrals given by equation (16) correspond to the Kowalewski exponents $\rho = 2$ and $\rho = 3$ respectively.

Finally, for $n = 4$ with the Cartan matrix

$$\{K_{\alpha\beta}\} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} \tag{18}$$

the equations of motion are

$$\begin{aligned} &\dot{p}_1 = q_1^2 \qquad \dot{p}_2 = q_2^2 \qquad \dot{p}_3 = q_3^2 \\ &\dot{q}_1 = q_1(p_1 - \tfrac{1}{2}p_2) \qquad \dot{q}_2 = q_2(-\tfrac{1}{2}p_1 + p_2 - \tfrac{1}{2}p_3) \\ &\dot{q}_2 = q_3(-\tfrac{1}{2}p_2 + p_3). \end{aligned} \tag{19}$$

Here the REDUCE package (Schwarz 1986a) finds the following three independent first integrals of respective degrees 2, 3 and 4:

$$\begin{aligned} h_1(\boldsymbol{p}, \boldsymbol{q}) &= p_1^2 + p_2^2 + p_3^2 + p_1p_2 - q_1^2 - q_2^2 - q_3^2 \\ h_2(\boldsymbol{p}, \boldsymbol{q}) &= p_1p_2(p_1 - p_2) + p_2p_3(p_2 - p_3) - q_1^2p_2 + q_2^2(p_1 - p_3) + q_3^2p_2 \\ h_3(\boldsymbol{p}, \boldsymbol{q}) &= p_1^4 + p_2^4 + p_3^4 - p_1p_2(2p_1^2 - 3p_1p_2 + 2p_2^2) - p_2p_3(2p_2^2 - 3p_2p_3 + 2p_3^2) \\ &\quad - 2q_1^2(p_1^2 - p_1p_2 + p_2^2) - 2q_2^2(p_1^2 + p_2^2 + p_3^2 - p_1p_2 - p_2p_3 - p_2p_3) \\ &\quad - 2q_3^2(p_2^2 - p_2p_3 + p_3^2) + 2q_2^2(q_1^2 + q_3^2) + q_1^4 + q_2^4 + q_3^4. \end{aligned} \tag{20}$$

The algebraic system for the c_i $(i = 1, \ldots, 6)$ has sixteen solutions now. For the main branch the values $c_1 = c_3 = -3$, $c_2 = -4$, $c_4^2 = c_6^2 = 3$ and $c_5^2 = 4$ are obtained which lead to the Kowalewski determinant (up to a constant factor)

$$\det(K_{ij} - \rho\delta_{ij}) = (\rho+3)(\rho+2)(\rho+1)(\rho-2)(\rho-3)(\rho-4). \tag{21}$$

The three first integrals given by equation (20) correspond to the Kowalewski exponents $\rho = 2$, $\rho = 3$ and $\rho = 4$ respectively.

In all cases discussed a consistent picture has been obtained by the two methods of investigation. Yoshida's theorem assures the possible existence of algebraic first integrals. Beyond that it is quite useful because it limits the order of the ansatz for an algebraic first integral with a non-vanishing gradient on solutions of equation (3). By direct search with the REDUCE package we have obtained the additional result that there does not exist a polynomial first integral of order up to and including six in both cases (16) and (19).

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